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AN ASYMPTOTIC THEORY FOR DYNAMIC RESPONSE OF ANISOTROPIC INHOMOGENEOUS AND LAMINATED PLATES

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Abstract—We develop an asymptotic theory for dynamic analysis of anisotropic inhomogeneous plates within the framework of three-dimensional elasticity. The inhomogeneities are considered to be in the thickness direction and the laminated plates belong to an important class of this type of inhomogeneous plates. Through nondimensionalization and introduction of the multiple time scales in the formulation, we obtain a uniform expansion of the field variables in even powers of a small plate parameter. The expansion yields an asymptotic solution valid regardless of the time span, whereas a straightforward expansion fails to produce convergent results. We show by successive integration that the equations for the asymptotic solution are of the same form as those in the classical laminated plate theory (CLT). While the asymptotic solution is no more difficult than the CLT solution, it is capable of yielding displacements and all the stress components in a consistent and systematic manner. Modifications to the lower-order solutions are made by eliminating the secular terms in the equations according to the method of multiple scales. The basic theory is illustrated by determining the free vibration characteristics of a symmetric cross-ply laminated plate.

1. INTRODUCTION

In a recent paper (Tarn and Wang, 1993), we developed an asymptotic theory, based on three-dimensional elasticity without *a priori* assumptions, for stress analysis of anisotropic inhomogeneous and laminated plates subject to thermomechanical loading. The inhomogeneities are considered to vary through the plate thickness, and laminated plates are important cases of inhomogeneous plates in which the elastic moduli are represented by piecewise constant functions through the thickness. In the case of laminated plates, the conditions of traction and displacement continuity between the lamina are inherently satisfied in the theory. There is no need to treat the interfacial continuity in particular. We showed that the equations for the asymptotic solution are precisely the governing equations with loading terms in the classical laminated plate theory (CLT). While the three-dimensional solution for a problem can be obtained in a systematic manner no more difficult than the CLT solution, the asymptotic solution converges rapidly and gives accurate results. In this paper we continue the study and extend the theory to dynamic analysis of anisotropic inhomogeneous and laminated plates.

There exist a variety of theories for dynamic response of plates. An assessment of various plate theories may be found in a review paper (Noor and Burton, 1989). It is well known that the effects of rotary inertia and transverse shear deformation are significant in the dynamic response of anisotropic laminated plates. The CLT does not give reliable results and extension of the theory to include these effects greatly complicates the formulation [e.g. Yang *et al.* (1966), Ashton and Whitney (1970), Whitney and Pagano (1970), Reddy (1982), Reddy and Phan (1985), Khdeir and Reddy (1989)]. Besides, any plate theory based on *a priori* assumptions regarding the variation of displacements through the thickness provides little information about the error estimation in the response predictions. Development of an asymptotic theory for motion of elastic plates was attempted earlier (Widera, 1970; Johnson and Widera, 1971). A straightforward expansion of all the field variables in the elasticity equations in powers of a small parameter was made, leading to asymptotic equations in rather cumbersome forms. Only the first-order equations were derived. The higher-order equations become too complicated to be meaningful even for isotropic materials. Their exposition is obscure to us in that it started by assuming the elastic moduli and the mass density of the material can be expanded in terms of the designated small parameter in the formulation. Nevertheless, it was shown that the first-order approximation yields the thin plate equations. No specific problem was solved in their papers to demonstrate the applicability of the theory. As will be shown in this paper, the use of straightforward expansion *does not* result in a valid solution for dynamic response of the plate.

In the present formulation we restrict ourselves to the motion of the plate in which the characteristic wavelength is of the same order as a linear dimension of the plate. It follows essentially the basic approach of our previous work. However, special attention must be paid to the nonlinear dependence of the transient response on the time variable. In the asymptotic solution we anticipate that the frequency spectrum associated with the higher-order effect will be different from that associated with the first-order approximation. As the solution of higher-order equations in a uniform asymptotic expansion must represent a small correction to the first-order response, a straightforward expansion of the displacement and stress components using only a single time scale *will not* yield a valid asymptotic solution. To obtain uniformly valid expansions regardless of the time span, it is necessary to employ special perturbation techniques that account for the nonlinear time dependence of the transient response. The method of multiple scales (Nayfeh, 1981) is adopted herein and proved to be effective. In the formulation different time scales are introduced, instead of determining the dynamic response as a function of a single time variable, we determine the field quantities as functions of different time scales so that their variations can be more closely examined. The situation is similar to observing the variations on the different time scales of a watch. Upon introducing multiple time scales in the expansion and eliminating secular terms at each level of asymptotic solution according to the method of multiple scales, we show that the present asymptotic theory not only provides a uniform expansion but also results in two-dimensional equations which include rotary inertia and shear deformation effects for the asymptotic solution. The equations are reduced to the CLT equations if these effects are neglected. Improvements in the asymptotic solution for the dynamic response can be made in a clear and systematic manner. As in our previous work, the expansions of the displacement and stress components are in powers of ε^2, where $\varepsilon = h/L$, $2h$ is the plate thickness and L is a typical in-plane plate dimension. Thus, the advantages in its elastostatic counterpart remain.

In Section 2 we present the relevant three-dimensional equations and the prescribed conditions for a general problem. The equations are recast into forms convenient for the subsequent analysis. In Section 3 nondimensionalization and multiple time scales are introduced in the asymptotic expansion. Successive integrations of the equations for the first and second order approximation are detailed in Section 4. It becomes clear in the exposition that a straightforward expansion does not provide a valid asymptotic solution for long times whereas the use of multiple scales does by eliminating the secular terms. To illustrate the basic theory, we present in Section 5 the solution for the motions of a simply-supported cross-ply symmetric laminated plate. The natural frequencies and the normal modes of free vibration are determined in a consistent manner. Application of the theory to general problems is discussed.

2. BASIC THREE-DIMENSIONAL EQUATIONS

We consider an inhomogeneous anisotropic plate of uniform thickness $2h$, having in each point one plane of elastic symmetry parallel to the midplane. Let us select a Cartesian coordinate system such that the plane $x_3 = 0$ coincides with the midplane of the plate and the x_3 axis is directed downward from the origin. Along the edge boundary of the plate, appropriate edge conditions are prescribed. On the lateral surface $x_3 = -h$, the transverse load $q(x_1, x_2, t)$ is prescribed. On the surface $x_3 = h$, the plate is free from external load.

The stress–displacement constitutive relations expressed in the chosen axes of the plate are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \\ u_{1,2}+u_{2,1} \end{Bmatrix}, \tag{1}$$

where the displacement components are denoted by u_1, u_2, u_3 and the commas denote differentiation with respect to the suffix variables. $\sigma_{11}, \sigma_{22}, \ldots, \sigma_{12}$ are the stress components. c_{ij} $(i, j = 1, 2, \ldots, 6)$ are the 13 elastic constants of the material with one plane of material symmetry. The material is assumed to be inhomogeneous through the plate thickness. Thus, $c_{ij} = c_{ij}(x_3)$. An important class of inhomogeneous plates is laminated plates, in which the "elastic constants" and mass density of the material are piecewise constant in the thickness direction.

The equations of motion are

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{2}$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = \rho \frac{\partial^2 u_2}{\partial t^2}, \tag{3}$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = \rho \frac{\partial^2 u_3}{\partial t^2}, \tag{4}$$

where $\rho = \rho(x_3)$ is the mass density of the inhomogeneous material.

Expressing the stresses in terms of the differential operators with respect to u_1, u_2, u_3 in (1) and eliminating σ_{11}, σ_{22}, σ_{12} by using (2), (3), we may recast the three-dimensional equations in such forms that on the left-hand side the differentiations are with respect to x_3, whereas on the right-hand side the differential operators are all expressed in terms of x_1, x_2, as given by

$$u_{3,3} = -[L_{13} \quad L_{23}] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + L_{33}\sigma_{33}, \tag{5}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_{,3} = -\begin{bmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} \end{bmatrix}^{\mathrm{T}} u_3 + \begin{bmatrix} s_{55} & s_{45} \\ s_{45} & s_{44} \end{bmatrix} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix}, \tag{6}$$

$$\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix}_{,3} = -\begin{bmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} L_{13} \\ L_{23} \end{Bmatrix} \sigma_{33} + \rho \frac{\partial^2}{\partial t^2} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \tag{7}$$

$$\sigma_{33,3} = -\begin{bmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} \end{bmatrix} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} + \rho \frac{\partial^2 u_3}{\partial t^2}, \tag{8}$$

where the superscript T denotes the transpose, and

$$L_{11} = \left[Q_{11} \frac{\partial^2}{\partial x_1^2} + 2Q_{16} \frac{\partial^2}{\partial x_1 \partial x_2} + Q_{66} \frac{\partial^2}{\partial x_2^2} \right],$$

$$L_{12} = \left[Q_{16} \frac{\partial^2}{\partial x_1^2} + (Q_{12}+Q_{66}) \frac{\partial^2}{\partial x_1 \partial x_2} + Q_{26} \frac{\partial^2}{\partial x_2^2} \right],$$

$$L_{22} = \left[Q_{66}\frac{\partial^2}{\partial x_1^2} + 2Q_{26}\frac{\partial^2}{\partial x_1 \partial x_2} + Q_{22}\frac{\partial^2}{\partial x_2^2} \right],$$

$$L_{13} = \left(c_{13}\frac{\partial}{\partial x_1} + c_{36}\frac{\partial}{\partial x_2} \right) \Big/ c_{33}, \quad L_{23} = \left(c_{36}\frac{\partial}{\partial x_1} + c_{23}\frac{\partial}{\partial x_2} \right) \Big/ c_{33},$$

$$L_{33} = 1/c_{33}, \quad \begin{bmatrix} s_{55} & s_{45} \\ s_{45} & s_{44} \end{bmatrix} = \begin{bmatrix} c_{55} & c_{45} \\ c_{45} & c_{44} \end{bmatrix}^{-1},$$

$$Q_{ij} = c_{ij} - c_{i3}c_{j3}/c_{33}, \quad (i, j = 1, 2, 6).$$

The in-plane stresses σ_{11}, σ_{22}, σ_{12} expressed in terms of u_1, u_2 and σ_{33} are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11}\frac{\partial}{\partial x_1} + Q_{16}\frac{\partial}{\partial x_2} & Q_{16}\frac{\partial}{\partial x_1} + Q_{12}\frac{\partial}{\partial x_2} \\ Q_{12}\frac{\partial}{\partial x_1} + Q_{26}\frac{\partial}{\partial x_2} & Q_{26}\frac{\partial}{\partial x_1} + Q_{22}\frac{\partial}{\partial x_2} \\ Q_{16}\frac{\partial}{\partial x_1} + Q_{66}\frac{\partial}{\partial x_2} & Q_{66}\frac{\partial}{\partial x_1} + Q_{26}\frac{\partial}{\partial x_2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{Bmatrix} c_{13}/c_{33} \\ c_{23}/c_{33} \\ c_{36}/c_{33} \end{Bmatrix} \sigma_{33}. \tag{9}$$

To facilitate subsequent analysis, the boundary conditions for the problem are written in matrix form as follows:

On the lateral surface the transverse load $q(x_1, x_2, t)$ is prescribed:

$$\begin{aligned} &[\sigma_{13} \quad \sigma_{23}] = [0, 0] && \text{on } x_3 = \pm h, \\ &\sigma_{33} = -q(x_1, x_2, t) && \text{on } x_3 = -h, \\ &\sigma_{33} = 0 && \text{on } x_3 = h. \end{aligned} \tag{10}$$

Along the edges Γ_σ tractions p_1, p_2, p_3 are prescribed:

$$\left. \begin{aligned} &\begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}, \\ &[n_1 \quad n_2] \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = p_3, \end{aligned} \right\} \text{on } \Gamma_\sigma, \tag{11}$$

in which n_1, n_2 denote the outward unit normal at a point along the edge.

Along the edges Γ_u displacements u_1^0, u_2^0, u_3^0 are prescribed:

$$u_1 = u_1^0, \quad u_2 = u_2^0, \quad u_3 = u_3^0, \quad \text{on } \Gamma_u. \tag{12}$$

In addition, the initial conditions for a transient response are prescribed by considering the displacements and their time derivatives throughout the plate in the initial state of the motion.

3. NONDIMENSIONALIZATION AND MULTIPLE SCALES

Let us restrict attention to the dynamic response of the plate in which the characteristic wavelength is of the same order as the linear dimension of the plate. As in its elastostatic counterpart we define the dimensionless coordinates, displacements and stresses as follows:

$$x = x_1/L, \quad y = x_2/L, \quad z = x_3/h,$$

$$u = u_1/h, \quad v = u_2/h, \quad w = u_3/L,$$

$$\sigma_x = \sigma_{11}/Q\varepsilon, \quad \sigma_y = \sigma_{22}/Q\varepsilon, \quad \sigma_{xy} = \sigma_{12}/Q\varepsilon,$$

$$\sigma_{xz} = \sigma_{13}/Q\varepsilon^2, \quad \sigma_{yz} = \sigma_{23}/Q\varepsilon^2, \quad \sigma_z = \sigma_{33}/Q\varepsilon^3, \tag{13}$$

in which $\varepsilon = h/L < 1$ is a dimensionless parameter, L denotes a typical inplane dimension of the plate, and $-1 < z < 1$. A specific elastic modulus or a reference uniform load which has the dimension of the elastic constants may be chosen as Q. To be specific, let $Q = c_{33}$.

As mentioned earlier, instead of determining the response as a function of t, we determine the field quantities as functions of different time scales. Furthermore, it is desirable to have an asymptotic expansion capable of yielding recurrence relations for all the displacement and stress components at each level of approximation. To this end, we introduce in the formulation the multiple dimensionless scales:

$$\tau_k = \frac{\varepsilon^{2k}}{L}\sqrt{\frac{c_{33}}{\rho_0}}\,t \quad (k = 0, 1, 2, \ldots), \tag{14}$$

in which ρ_0 is a reference mass density.

Because ε is a small parameter, τ_0 represents a fast scale, τ_1 represents a slower scale, τ_2 an even slower scale, and so on. Upon introducing (13), (14) in (5)–(9) and using the chain rule of differentiation, the dimensionless equations can be written as

$$w_{,z} = -\varepsilon^2[l_{13} \quad l_{23}]\begin{Bmatrix} u \\ v \end{Bmatrix} + \varepsilon^4 \sigma_z, \tag{15}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix}_{,z} = -\begin{bmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} \end{bmatrix}^{\mathrm{T}} w + \varepsilon^2 \begin{bmatrix} \tilde{s}_{55} & \tilde{s}_{45} \\ \tilde{s}_{45} & \tilde{s}_{44} \end{bmatrix}\begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}, \tag{16}$$

$$\begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}_{,z} = -\begin{bmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} - \varepsilon^2 \begin{Bmatrix} l_{13} \\ l_{23} \end{Bmatrix}\sigma_z + \rho_1\left(\frac{\partial^2}{\partial \tau_0^2} + 2\varepsilon^2 \frac{\partial^2}{\partial \tau_1 \partial \tau_0} + \ldots\right)\begin{Bmatrix} u \\ v \end{Bmatrix}, \tag{17}$$

$$\sigma_{z,z} = -\begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{bmatrix}\begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} + \rho_2\left(\frac{\partial^2}{\partial \tau_0^2} + 2\varepsilon^2 \frac{\partial^2}{\partial \tau_1 \partial \tau_0} + \ldots\right) w \tag{18}$$

and

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} l_{14} & l_{24} \\ l_{15} & l_{25} \\ l_{16} & l_{26} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} + \varepsilon^2 \begin{Bmatrix} l_{34} \\ l_{35} \\ l_{36} \end{Bmatrix}\sigma_z, \tag{19}$$

where

$$\rho_1 = \rho/\rho_0, \quad \rho_2 = \frac{L^2 \rho}{h^2 \rho_0},$$

$$l_{11} = \left[\tilde{Q}_{11}\frac{\partial^2}{\partial x^2} + 2\tilde{Q}_{16}\frac{\partial^2}{\partial x\,\partial y} + \tilde{Q}_{66}\frac{\partial^2}{\partial y^2}\right],$$

$$l_{12} = \left[\tilde{Q}_{16}\frac{\partial^2}{\partial x^2} + (\tilde{Q}_{12} + \tilde{Q}_{66})\frac{\partial^2}{\partial x\,\partial y} + \tilde{Q}_{26}\frac{\partial^2}{\partial y^2}\right],$$

$$l_{22} = \left[\tilde{Q}_{66}\frac{\partial^2}{\partial x^2} + 2\tilde{Q}_{26}\frac{\partial^2}{\partial x\,\partial y} + \tilde{Q}_{22}\frac{\partial^2}{\partial y^2}\right],$$

$$l_{13} = \left(c_{13}\frac{\partial}{\partial x} + c_{36}\frac{\partial}{\partial y}\right)\Big/c_{33}, \quad l_{23} = \left(c_{36}\frac{\partial}{\partial x} + c_{23}\frac{\partial}{\partial y}\right)\Big/c_{33},$$

$$l_{14} = \left(\tilde{Q}_{11}\frac{\partial}{\partial x} + \tilde{Q}_{16}\frac{\partial}{\partial y}\right), \quad l_{24} = \left(\tilde{Q}_{16}\frac{\partial}{\partial x} + \tilde{Q}_{12}\frac{\partial}{\partial y}\right),$$

$$l_{15} = \left(\tilde{Q}_{12}\frac{\partial}{\partial x} + \tilde{Q}_{26}\frac{\partial}{\partial y}\right), \quad l_{25} = \left(\tilde{Q}_{26}\frac{\partial}{\partial x} + \tilde{Q}_{22}\frac{\partial}{\partial y}\right),$$

$$l_{16} = \left(\tilde{Q}_{16}\frac{\partial}{\partial x} + \tilde{Q}_{66}\frac{\partial}{\partial y}\right), \quad l_{26} = \left(\tilde{Q}_{66}\frac{\partial}{\partial x} + \tilde{Q}_{26}\frac{\partial}{\partial y}\right),$$

$$l_{34} = c_{13}/c_{33}, \quad l_{35} = c_{23}/c_{33}, \quad l_{36} = c_{36}/c_{33},$$

$$\tilde{Q}_{ij} = Q_{ij}/c_{33}\ (i,j = 1,2,6), \quad \tilde{s}_{ij} = s_{ij}c_{33} \quad (i,j = 4,5).$$

We seek a uniform expansion of all the displacement and the stress components in powers of ε^2 in the form:

$$f(x,y,z,\tau_k;\varepsilon) = f_{(0)}(x,y,z,\tau_k) + \varepsilon^2 f_{(1)}(x,y,z,\tau_k) + \varepsilon^4 f_{(2)}(x,y,z,\tau_k) + \dots \quad (20)$$

Upon substituting (20) into (15)–(19) and collecting coefficients of equal powers of ε, we obtain the following sets of equations:
Order ε^0:

$$w_{(0),z} = 0, \quad (21)$$

$$\mathbf{u}_{(0),z} = -\mathbf{D}^{\mathrm{T}} w_{(0)}, \quad (22)$$

$$\boldsymbol{\sigma}_{s(0),z} = -\mathbf{L}_1\mathbf{u}_{(0)} + \rho_1\frac{\partial^2}{\partial\tau_0^2}\mathbf{u}_{(0)}, \quad (23)$$

$$\boldsymbol{\sigma}_{z(0),z} = -\mathbf{D}\boldsymbol{\sigma}_{s(0)} + \rho_2\frac{\partial^2}{\partial\tau_0^2}w_{(0)} \quad (24)$$

and

$$\boldsymbol{\sigma}_{p(0)} = \mathbf{L}_3\mathbf{u}_{(0)}. \quad (25)$$

Order ε^2:

$$w_{(1),z} = -\mathbf{L}_2\mathbf{u}_{(0)}, \quad (26)$$

$$\mathbf{u}_{(1),z} = -\mathbf{D}^{\mathrm{T}} w_{(1)} + \mathbf{S}\boldsymbol{\sigma}_{s(0)}, \quad (27)$$

$$\boldsymbol{\sigma}_{s(1),z} = -\mathbf{L}_1\mathbf{u}_{(1)} - \mathbf{L}_2^{\mathrm{T}}\boldsymbol{\sigma}_{z(0)} + \rho_1\frac{\partial^2}{\partial\tau_0^2}\mathbf{u}_{(1)} + 2\rho_1\frac{\partial^2}{\partial\tau_1\,\partial\tau_0}\mathbf{u}_{(0)}, \quad (28)$$

$$\sigma_{z(1),z} = -\mathbf{D}\boldsymbol{\sigma}_{s(1)} + \rho_2\frac{\partial^2}{\partial\tau_0^2}w_{(1)} + 2\rho_2\frac{\partial^2}{\partial\tau_1\,\partial\tau_0}w_{(0)} \quad (29)$$

and

$$\boldsymbol{\sigma}_{p(1)} = \mathbf{L}_3\mathbf{u}_{(1)} + \mathbf{L}_4\boldsymbol{\sigma}_{z(0)}, \tag{30}$$

where

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \boldsymbol{\sigma}_s = \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix}, \quad \boldsymbol{\sigma}_p = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{bmatrix}, \quad \mathbf{L}_1 = \begin{bmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{bmatrix},$$

$$\mathbf{L}_2 = [l_{13} \quad l_{23}], \quad \mathbf{L}_3 = \begin{bmatrix} l_{14} & l_{24} \\ l_{15} & l_{25} \\ l_{16} & l_{26} \end{bmatrix}, \quad \mathbf{L}_4 = \begin{bmatrix} l_{34} \\ l_{35} \\ l_{36} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \tilde{s}_{55} & \tilde{s}_{45} \\ \tilde{s}_{45} & \tilde{s}_{44} \end{bmatrix}.$$

Higher-order equations can be written out if necessary. When the field variables are independent of time, (21)–(30) are reduced to the recurrence equations for elastostatics.

The associated dimensionless boundary conditions are:

Order ε^0:

$$\boldsymbol{\sigma}_{s(0)} = 0 \quad \text{on } z = \pm 1, \tag{31}$$

$$\sigma_{z(0)} = -\tilde{q} \quad \text{on } z = -1, \tag{32}$$

$$\sigma_{z(0)} = 0 \quad \text{on } z = 1, \tag{33}$$

$$\begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix} \mathbf{L}_3\mathbf{u}_{(0)} = \begin{Bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{Bmatrix} \quad \text{on } \Gamma_\sigma, \tag{34}$$

$$[n_1 \quad n_2]\boldsymbol{\sigma}_{s(0)} = \tilde{p}_3, \tag{35}$$

$$u_{(0)} = u^0, \quad v_{(0)} = v^0, \quad w_{(0)} = w^0 \quad \text{on } \Gamma_u. \tag{36}$$

Order ε^{2k}, $(k = 1, 2, 3, \ldots)$:

$$\boldsymbol{\sigma}_{s(k)} = 0 \quad \text{on } z = \pm 1, \tag{37}$$

$$\sigma_{z(k)} = 0 \quad \text{on } z = \pm 1, \tag{38}$$

$$\begin{bmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{bmatrix} (\mathbf{L}_3\mathbf{u}_{(k)} + \mathbf{L}_4\sigma_{z(k-1)}) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{on } \Gamma_\sigma, \tag{39}$$

$$[n_1 \quad n_2]\,\boldsymbol{\sigma}_{s(k)} = 0, \tag{40}$$

$$u_{(k)} = v_{(k)} = w_{(k)} = 0 \quad \text{on } \Gamma_u, \tag{41}$$

where

$$\tilde{q} = q/Q\varepsilon^3, \quad \tilde{p}_k = p_k/Q\varepsilon, \quad (k = 1, 2), \quad \text{and} \quad \tilde{p}_3 = p_3/Q\varepsilon^2,$$
$$u^0 = u_1^0/h, \quad v^0 = u_2^0/h, \quad w^0 = u_3^0/L.$$

4. ASYMPTOTIC INTEGRATION

The resulting differential equations can be integrated successively with respect to z to determine the solution for a problem. Accordingly, we obtain from (21):

$$w_{(0)} = w_0(x, y, \tau_0, \tau_1, \ldots). \tag{42}$$

From (22) we have

$$\mathbf{u}_{(0)} = -z\mathbf{D}^{t}w_0 + \mathbf{u}_0(x, y, \tau_0, \tau_1, \ldots). \tag{43}$$

Integrating (23) and (24), we obtain

$$\boldsymbol{\sigma}_{s(0)} = \int_{-1}^{z} \mathbf{L}_1(\eta\mathbf{D}^{\mathrm{T}}w_0 - \mathbf{u}_0)\,\mathrm{d}\eta - \frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z} \rho_1(\eta\mathbf{D}^{\mathrm{T}}w_0 - \mathbf{u}_0)\,\mathrm{d}\eta, \tag{44}$$

$$\sigma_{z(0)} = -\mathbf{D}\int_{-1}^{z}\left\{\int_{-1}^{z}\mathbf{L}_1(z\mathbf{D}^{\mathrm{T}}w_0 - \mathbf{u}_0)\,\mathrm{d}z\right\}\mathrm{d}z + \frac{\partial^2}{\partial\tau_0^2}\mathbf{D}\int_{-1}^{z}\left\{\int_{-1}^{z}\rho_1(z\mathbf{D}^{\mathrm{T}}w_0 - \mathbf{u}_0)\,\mathrm{d}z\right\}\mathrm{d}z + \left(\int_{-1}^{z}\rho_2\,\mathrm{d}z\right)\frac{\partial^2 w_0}{\partial\tau_0^2} - \tilde{q}$$

$$= -\mathbf{D}\int_{-1}^{z}(z-\eta)\mathbf{L}_1(\eta\mathbf{D}^{\mathrm{T}}w_0 - \mathbf{u}_0)\,\mathrm{d}\eta + \frac{\partial^2}{\partial\tau_0^2}\mathbf{D}\int_{-1}^{z}(z-\eta)\rho_1(\eta\mathbf{D}^{\mathrm{T}}w_0 - \mathbf{u}_0)\,\mathrm{d}\eta + \left(\int_{-1}^{z}\rho_2\,\mathrm{d}\eta\right)\frac{\partial^2 w_0}{\partial\tau_0^2} - \tilde{q}, \tag{45}$$

in reducing the double integral to the single integral we have used the integration by parts, and $\rho_i = \rho_i(\eta)$, $l_{ij} = l_{ij}(\eta)$, $(i, j = 1, 2)$ in $\mathbf{L}_1$.

The integration functions u_0, v_0, w_0 in these expressions are determined from the boundary conditions. The lateral conditions on $z = -1$ are identically satisfied by (44), (45). The boundary condition $[\sigma_{xz}, \sigma_{yz}]_{(0)} = [0, 0]$ on $z = 1$ gives:

$$\int_{-1}^{1}\mathbf{L}_1(\mathbf{u}_0 - \eta\mathbf{D}^{\mathrm{T}}w_0)\,\mathrm{d}\eta = \frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{1}\rho_1(\mathbf{u}_0 - \eta\mathbf{D}^{\mathrm{T}}w_0)\,\mathrm{d}\eta. \tag{46}$$

Performing the simple operation, we find that (46) becomes

$$A_{11}u_{0,xx} + 2A_{16}u_{0,xy} + A_{66}u_{0,yy} + A_{16}v_{0,xx} + (A_{12}+A_{66})v_{0,xy} + A_{26}v_{0,yy} - B_{11}w_{0,xxx} - 3B_{16}w_{0,xxy} - (B_{12}+2B_{66})w_{0,xyy} - B_{26}w_{0,yyy} = I_{10}\frac{\partial^2 u_0}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}(w_{0,x}), \tag{47}$$

$$A_{16}u_{0,xx} + (A_{12}+A_{66})u_{0,xy} + A_{26}u_{0,yy} + A_{66}v_{0,xx} + 2A_{26}v_{0,xy} + A_{22}v_{0,yy} - B_{16}w_{0,xxx} - (B_{12}+2B_{66})w_{0,xxy} - 3B_{26}w_{0,xyy} - B_{22}w_{0,yyy} = I_{10}\frac{\partial^2 v_0}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}(w_{0,y}), \tag{48}$$

where

$$I_{10} = \int_{-1}^{1}\rho_1\,dz, \quad I_{11} = \int_{-1}^{1}\rho_1 z\,\mathrm{d}z, \quad A_{ij} = \int_{-1}^{1}\tilde{Q}_{ij}\,\mathrm{d}z, \quad B_{ij} = \int_{-1}^{1}\tilde{Q}_{ij}z\,\mathrm{d}z, (i, j = 1, 2, 6).$$

The boundary condition $\sigma_{z(0)} = 0$ on $z = 1$ gives

$$\mathbf{D}\int_{-1}^{1}(1-z)\mathbf{L}_1(\mathbf{u}_0-z\mathbf{D}^{\mathrm{T}}w_0)\,\mathrm{d}z = \mathbf{D}\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{1}(1-z)\rho_1(\mathbf{u}_0-z\mathbf{D}^{\mathrm{T}}w_0)\,\mathrm{d}z$$
$$-\left(\int_{-1}^{1}\rho_2\,\mathrm{d}z\right)\frac{\partial^2 w_0}{\partial\tau_0^2}+\tilde{q}. \quad (49)$$

By using (46) in (49), this equation can be written explicitly as

$$D_{11}w_{0,xxxx}+4D_{16}w_{0,xxxy}+2(D_{12}+2D_{66})w_{0,xxyy}+4D_{26}w_{0,xyyy}+D_{22}w_{0,yyyy}-B_{11}u_{0,xxx}$$
$$-3B_{16}u_{0,xxy}-(B_{12}+2B_{66})u_{0,xyy}-B_{26}u_{0,yyy}-B_{16}v_{0,xxx}-(B_{12}+2B_{66})v_{0,xxy}-3B_{26}v_{0,xyy}$$
$$-\mathrm{B}_{22}v_{0,yyy}=\tilde{q}-I_{20}\frac{\partial^2 w_0}{\partial\tau_0^2}+I_{12}\frac{\partial^2}{\partial\tau_0^2}(w_{0,xx}+w_{0,yy})-I_{11}\frac{\partial^2}{\partial\tau_0^2}(u_{0,x}+v_{0,y}), \quad (50)$$

where

$$I_{12}=\int_{-1}^{1}\rho_1 z^2\,\mathrm{d}z,\quad I_{20}=\int_{-1}^{1}\rho_2\,\mathrm{d}z,\quad D_{ij}=\int_{-1}^{1}\tilde{Q}_{ij}z^2\,\mathrm{d}z,\quad (i,j=1,2,6).$$

The terms associated with I_{11} in (47) and (48) represent the effect of shear deformation on the in-plane motion, the term with I_{11} in (50) represents the effect of in-plane deformation on the flexural motion. The term with I_{12} is the flexural rotary inertia. If one starts with the displacement field according to Kirchhoff's thin-plate theory without omitting the terms of rotary inertia and shear deformation effects, it can be shown in a straightforward manner that (47), (48) and (50) are exactly the resulting governing equations for the displacements. Neglecting these effects, the equations are reduced to the CLT equations for a vibration problem [see, e.g. Ashton and Whitney (1970)]. When the inhomogeneities are symmetric about the midplane as for the case with symmetric laminated plates, $B_{ij}=0$ and $I_{11}=0$, it can be seen from the first-order equations that the in-plane motion and flexural motion are uncoupled. The solution of (47), (48), (50) must be supplemented with the edge conditions (34)–(36) and the initial conditions. Once u_0, v_0, w_0 are determined, the displacements and all the stress components of the first-order approximation can be obtained using (42), (45) and (25).

Carrying on the solution to order ε^2, we obtain

$$w_{(1)}=w_1(x,y,\tau_0,\tau_1,\ldots)+\varphi_1(x,y,z,\tau_0,\tau_1,\ldots), \quad (51)$$

$$\mathbf{u}_{(1)}=\mathbf{u}_1(x,y,\tau_0,\tau_1,\ldots)-z\mathbf{D}^{\mathrm{T}}w_1+\boldsymbol{\phi}_1(x,y,z,\tau_0,\tau_1,\ldots), \quad (52)$$

$$\boldsymbol{\sigma}_{s(1)}=-\int_{-1}^{z}\mathbf{L}_1(\mathbf{u}_1-\eta\mathbf{D}^{\mathrm{T}}w_1)\,\mathrm{d}\eta+\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1(\mathbf{u}_1-\eta\mathbf{D}^{\mathrm{T}}w_1)\,\mathrm{d}\eta+\mathbf{f}_1(x,y,z,\tau_0,\tau_1,\ldots), \quad (53)$$

$$\sigma_{z(1)}=\mathbf{D}\int_{-1}^{z}(z-\eta)\mathbf{L}_1(\mathbf{u}_1-\eta\mathbf{D}^{\mathrm{T}}w_1)\mathrm{d}\eta-\mathbf{D}\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}(z-\eta)\rho_1$$
$$\times(\mathbf{u}_1-\eta\mathbf{D}^{\mathrm{T}}w_1)\,\mathrm{d}\eta+\left(\int_{-1}^{z}\rho_2\,\mathrm{d}\eta\right)\frac{\partial^2 w_1}{\partial\tau_0^2}+q_1(x,y,z,\tau_0,\tau_1,\ldots), \quad (54)$$

in which

$$\varphi_1 = -\int_0^z \mathbf{L}_2\mathbf{u}_{(0)}\,\mathrm{d}z, \qquad \mathbf{u}_1 = [u_1 \quad v_1]^{\mathrm{T}},$$

$$\boldsymbol{\phi}_1 = [\phi_{11} \quad \phi_{21}]^{\mathrm{T}} = \int_0^z (\mathbf{S}\boldsymbol{\sigma}_{s(0)} - \mathbf{D}^{\mathrm{T}}\varphi_1)\,\mathrm{d}\eta,$$

$$\mathbf{f}_1 = [f_{11} \quad f_{21}]^{\mathrm{T}} = -\int_{-1}^z (\mathbf{L}_1\boldsymbol{\phi}_1 + \mathbf{L}_2^{\mathrm{T}}\sigma_{z(0)})\,\mathrm{d}\eta + \frac{\partial^2}{\partial\tau_0^2}\int_{-1}^z \rho_1\boldsymbol{\phi}_1\,\mathrm{d}\eta + 2\frac{\partial^2}{\partial\tau_1\,\partial\tau_0}\int_{-1}^z \rho_1\mathbf{u}_{(0)}\,\mathrm{d}\eta,$$

$$q_1 = -\mathbf{D}\int_{-1}^z \mathbf{f}_1\,\mathrm{d}\eta + \frac{\partial^2}{\partial\tau_0^2}\int_{-1}^z \rho_2\varphi_1\,\mathrm{d}\eta + 2\frac{\partial^2}{\partial\tau_1\,\partial\tau_0}\int_{-1}^z \rho_2 w_0\,\mathrm{d}\eta.$$

Note that the lower limits in the integrals of φ_1 and ϕ_1 have been deliberately chosen as zero so that $\varphi_1 = \boldsymbol{\phi}_1 = 0$ at $z = 0$, thus the displacements on the midplane can be more conveniently examined. By using the boundary conditions (37), (38) to determine u_1, v_1, w_1 as before, it is easy to see that the boundary conditions on $z = -1$ are identically satisfied, and the condition $[\sigma_{xz}, \sigma_{yz}]_{(1)} = [0, 0]$ on $z = 1$ leads to

$$\begin{aligned} A_{11}u_{1,xx} + 2A_{16}u_{1,xy} + A_{66}u_{1,yy} + A_{16}v_{1,xx} + (A_{12}+A_{66})v_{1,xy} + A_{26}v_{1,yy} - B_{11}w_{1,xxx} \\ - 3B_{16}w_{1,xxy} - (B_{12}+2B_{66})w_{1,xyy} - B_{26}w_{1,yyy} \\ = I_{10}\frac{\partial^2 u_1}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}(w_{1,x}) + f_{11}(x, y, 1, \tau_0, \tau_1, \ldots), \end{aligned} \tag{55}$$

$$\begin{aligned} A_{16}u_{1,xx} + (A_{12}+A_{66})u_{1,xy} + A_{26}u_{1,yy} + A_{66}v_{1,xx} + 2A_{26}v_{1,xy} + A_{22}v_{1,yy} - B_{16}w_{1,xxx} \\ - (B_{12}+2B_{66})w_{1,xxy} - 3B_{26}w_{1,xyy} - B_{22}w_{1,yyy} \\ = I_{10}\frac{\partial^2 v_1}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}(w_{1,y}) + f_{21}(x, y, 1, \tau_0, \tau_1, \ldots). \end{aligned} \tag{56}$$

Using (54) in the condition $\sigma_{z(1)} = 0$ on $z = 1$ leads to

$$\begin{aligned} D_{11}w_{1,xxxx} + 4D_{16}w_{1,xxxy} + 2(D_{12}+2D_{66})w_{1,xxyy} + 4D_{26}w_{1,xyyy} + D_{22}w_{1,yyyy} \\ - B_{11}u_{1,xxx} - 3B_{16}u_{1,xxy} - (B_{12}+2B_{66})u_{1,xyy} - B_{26}u_{1,yyy} - B_{16}v_{1,xxx} - (B_{12}+2B_{66})v_{1,xxy} \\ - 3B_{26}v_{1,xyy} - B_{22}v_{1,yyy} \\ = -I_{20}\frac{\partial^2 w_1}{\partial\tau_0^2} + I_{12}\frac{\partial^2}{\partial\tau_0^2}(w_{1,xx} + w_{1,yy}) - I_{11}\frac{\partial^2}{\partial\tau_0^2}(u_{1,x} + v_{1,y}) - q_1(x, y, 1, \tau_0, \tau_1, \ldots) \\ - f_{11,1}(x, y, 1, \tau_0, \tau_1, \ldots) - f_{21,2}(x, y, 1, \tau_0, \tau_1, \ldots). \end{aligned} \tag{57}$$

Equations (55)–(57) are of the same forms as the first-order equations except with the added equivalent "forcing" terms f_{11}, f_{21} and q_1. Had we not introduced the multiple time scales $\tau_1, \tau_2, \ldots,$ in the formulation, we would have ended up with forcing functions, completely defined by the first-order solution, having exactly the same frequencies as those of the first-order natural frequency. This would either induce resonance so that the modifications become unbounded in magnitude, or yield the ε^2 solution that contains secular terms that are not small for long periods of time. Consequently, the asymptotic expansion would break down. For the expansion to be uniform and valid regardless of the time span, the higher-order corrections must be free of secular terms. Obviously, the presence of f_{11}, f_{21} and q_1 is the source of the secular terms in the general case. Thus, for uniform expansion we must distinguish from f_{11}, f_{21} and q_1 the terms that show the same frequencies as the considered modes of vibration. These terms must be eliminated. As the functional dependence of each quantity in f_{11}, f_{21} and q_1 upon τ_0 is known from the first-order solution, setting the coefficients of the functions that cause secular terms for the second order corrections equal to zero will result in a system of quasi-ordinary differential equations with independent variables $\tau_1, \tau_2, \ldots$, from which the dependence of the field variables on τ_1 can be obtained. It follows that the modification to the leading-order solution can be determined. We remark that the functional dependence of the field variables upon $\tau_2, \tau_3, \ldots$, is as yet unknown at the second-order level of approximation; it is determined in a similar fashion at the subsequent levels by eliminating the secular terms as one proceeds to the higher order.

While the solution at each level of approximation is no more difficult than the corresponding CLT solution, the present theory is capable of providing all the three-dimensional stress and displacement components for a problem and the asymptotic solution converges according to the powers of ε^2. The theory is best illustrated by applying it to a specific example.

5. ILLUSTRATIVE EXAMPLE

The motion of a cross-ply symmetric rectangular laminated plate is considered. The plate is composed of orthotropic layers with material axes coincident with the plate axes. With this lamination scheme, in addition to

$$B_{ij} = I_{11} = 0, \text{we have } A_{16} = A_{26} = D_{16} = D_{26} = Q_{16} = Q_{26} = 0.$$

We assume that the field quantities have been made dimensionless according to (13) for simplicity. The edges of the plate are simply supported with the boundary conditions:

$$v = w = 0, \quad \sigma_x = 0 \quad \text{on } x = 0, a, \tag{58}$$

$$u = w = 0, \quad \sigma_y = 0 \quad \text{on } y = 0, b. \tag{59}$$

As an illustration, let us consider the free motion. When the lateral transverse load $\tilde{q} = 0$, solution of (47), (48) and (50) for order ε^0 can be obtained by letting

$$u_0 = U_0 \cos\alpha x \sin\beta y \cos(\omega_{mn}\tau_0 + \delta_{mn}), \tag{60}$$

$$v_0 = V_0 \sin\alpha x \cos\beta y \cos(\omega_{mn}\tau_0 + \delta_{mn}), \tag{61}$$

$$w_0 = W_0 \sin\alpha x \sin\beta y \cos(\omega_{mn}\tau_0 + \delta_{mn}), \tag{62}$$

where $\alpha = m\pi/a$, $\beta = n\pi/b$. ω_{mn} are the circular frequencies of the motions. U_0, V_0 and W_0 are the amplitudes. The phase angles δ_{mn} are independent of τ_0 and are as yet undetermined functions of the time scales $\tau_1, \tau_2, \ldots$.

Upon substituting (60)–(62) in (47), (48) and (50), we have

$$(A_{11}\alpha^2 + A_{66}\beta^2 - I_{10}\omega_{mn}^2)U_0 + (A_{12} + A_{66})\alpha\beta V_0 = 0, \tag{63}$$

$$(A_{12} + A_{66})\alpha\beta U_0 + (A_{66}\alpha^2 + A_{22}\beta^2 - I_{10}\omega_{mn}^2)V_0 = 0, \tag{64}$$

$$\{D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4 - [I_{20} + I_{12}(\alpha^2 + \beta^2)]\omega_{mn}^2\} W_0 = 0. \tag{65}$$

Equations (63), (64) have a nontrivial solution only if

$$\begin{vmatrix} A_{11}\alpha^2+A_{66}\beta^2-I_{10}\omega_{mn}^2 & (A_{12}+A_{66})\alpha\beta \\ (A_{12}+A_{66})\alpha\beta & A_{66}\alpha^2+A_{22}\beta^2-I_{10}\omega_{mn}^2 \end{vmatrix} = 0, \tag{66}$$

which yields two positive roots for the first-order approximation of the natural frequencies for the in-plane motion.

From (65) the first-order frequencies for the flexural motion are given by

$$\omega_{mn} = [D_{11}\alpha^4+2(D_{12}+2D_{66})\alpha^2\beta^2+D_{22}\beta^4]^{1/2}/[I_{20}+I_{12}(\alpha^2+\beta^2)]^{1/2}. \tag{67}$$

Evidently the flexural motion and in-plane motion are not coupled at this level of approximation. The first-order frequencies of the flexural motion coincide with the CLT results [see, e.g. Aston and Whitney (1970)] if neglecting the rotary inertia ($I_{12} = 0$). For subsequent analysis we denote the frequencies and phase angles of the flexural motion and in-plane motion by ω^{f}_{mn}, $\omega^{(1)}_{mn}$, $\omega^{(2)}_{mn}$, and δ^{f}_{mn}, $\delta^{(1)}_{mn}$, $\delta^{(2)}_{mn}$, respectively. To each frequency there corresponds a normal mode of motion which is determined by using (60)–(65). In the following the motion corresponding to ω^{f}_{mn} is given in detail.

Using (60)–(62) in (42)–(45) and (25), we obtain the first-order solution corresponding to ω^{f}_{mn} as

$$u_{(0)} = -\alpha W_0 z \cos\alpha x \sin\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0+\delta^{\mathrm{f}}_{mn}), \tag{68}$$

$$v_{(0)} = -\beta W_0 z \sin\alpha x \cos\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0+\delta^{\mathrm{f}}_{mn}), \tag{69}$$

$$w_{(0)} = W_0 \sin\alpha x \sin\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0+\delta^{\mathrm{f}}_{mn}), \tag{70}$$

$$\sigma_{x(0)} = (\tilde{Q}_{11}\alpha^2+\tilde{Q}_{12}\beta^2) W_0 z \sin\alpha x \sin\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0+\delta^{\mathrm{f}}_{mn}), \tag{71}$$

$$\sigma_{y(0)} = (\tilde{Q}_{12}\alpha^2+\tilde{Q}_{22}\beta^2) W_0 z \sin\alpha x \sin\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0+\delta^{\mathrm{f}}_{mn}), \tag{72}$$

$$\sigma_{xy(0)} = -2\tilde{Q}_{66}\alpha\beta W_0 z \cos\alpha x \cos\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0+\delta^{\mathrm{f}}_{mn}), \tag{73}$$

$$\sigma_{xz(0)} = \sigma_{xz0}(z) W_0 \cos\alpha x \sin\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0+\delta^{\mathrm{f}}_{mn}), \tag{74}$$

$$\sigma_{yz(0)} = \sigma_{yz0}(z) W_0 \sin\alpha x \cos\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0+\delta^{\mathrm{f}}_{mn}), \tag{75}$$

$$\sigma_{z(0)} = \sigma_{z0}(z) W_0 \sin\alpha x \sin\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0+\delta^{\mathrm{f}}_{mn}), \tag{76}$$

where

$$\sigma_{xz0}(z) = -\int_{-1}^{z} [\tilde{Q}_{11}\alpha^3+(\tilde{Q}_{12}+2\tilde{Q}_{66})\alpha\beta^2-\rho_1\alpha(\omega^{\mathrm{f}}_{mn})^2]\eta\,\mathrm{d}\eta,$$

$$\sigma_{yz0}(z) = -\int_{-1}^{z} [(\tilde{Q}_{12}+2\tilde{Q}_{66})\alpha^2\beta+\tilde{Q}_{22}\beta^3-\rho_1\beta(\omega^{\mathrm{f}}_{mn})^2]\eta\,\mathrm{d}\eta,$$

$$\sigma_{z0}(z) = -\int_{-1}^{z} \eta(z-\eta)[\tilde{Q}_{11}\alpha^4+2\alpha^2\beta^2(\tilde{Q}_{12}+2\tilde{Q}_{66})+\tilde{Q}_{22}\beta^4 - \rho_1(\alpha^2+\beta^2)(\omega^{\mathrm{f}}_{mn})^2]\,\mathrm{d}\eta - \left(\int_{-1}^{z}\rho_2\,\mathrm{d}\eta\right)(\omega^{\mathrm{f}}_{mn})^2.$$

It can be verified that the edge conditions (58), (59) are exactly satisfied. If we stop here, δ^{f}_{mn}, may be considered constant. The results of the first-order approximation are accurate to $O(\varepsilon^2)$.

To obtain corrections to the first-order solution, we proceed to order ε^2. Upon substituting (68)–(76) in (51)–(54), we obtain at $z = 1$:

$$f_{11} = k_1 W_0 \cos\alpha x \sin\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn}), \tag{77}$$

$$f_{21} = k_2 W_0 \sin\alpha x \cos\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn}), \tag{78}$$

$$q_1 = \left(k_3 \frac{\partial \delta^{\mathrm{f}}_{mn}}{\partial \tau_1} + k_4\right) W_0 \sin\alpha x \sin\beta y \cos(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn}), \tag{79}$$

where k_i ($i = 1, 2, 3, 4$) and the relevant functions for the ε^2 solution are given in the Appendix.

Substituting for f_{11}, f_{21} and q_1 from (77)–(79) into (55)–(57), we find that the forcing term in (57) has ω^{f}_{mn} as its frequency. If we were to use it in the solution, the time dependence of the particular solution would be of the form $f(\tau_0)\cos(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn}) + g(\tau_0)\sin(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn})$, where f and g are functions of τ_0. Then the corrections are small only for restrictive values of τ_0. Since τ_0 is arbitrary, the corrections inevitably may become large compared with the first-order solution. This is not admissible for a uniform expansion unless the secular terms are eliminated. Inspecting (77)–(79), we find that $q_1 + f_{11,1} + f_{21,2}$ produces secular terms in the solution whereas f_{11} and f_{21} do not. Setting $q_1 + f_{11,1} + f_{21,2} = 0$ at $z = 1$ in (57), we have

$$k_3 \frac{\partial \delta^{\mathrm{f}}_{mn}}{\partial \tau_1} + k_4 - \alpha k_1 - \beta k_2 = 0. \tag{80}$$

The solution of (80) is

$$\delta^{\mathrm{f}}_{mn} = [(\alpha k_1 + \beta k_2 - k_4)/k_3]\tau_1 + \tilde{\delta}_{mn}(\tau_2, \tau_3, \ldots). \tag{81}$$

It is worth mentioning that had we not introduced the multiple scales $\tau_1, \tau_2, \ldots$, in addition to τ_0 in the formulation we would have had in q_1 only the k_4 term. Then it is impossible to avoid the secular terms and the expansion is doomed to failure.

Substituting (81) in (72)–(76), bearing in mind that $\tau_1 = \varepsilon^2\tau_0$, we find that all the field quantities are now time-dependent functions of $\cos\{[\omega^{\mathrm{f}}_{mn} + \varepsilon^2(\alpha k_1 + \beta k_2 - k_4)/k_3]\tau_0 + \tilde{\delta}_{mn}\}$. Thus, it may be deduced that the natural frequency at the ε^2 level of approximation is modified to $\omega^{\mathrm{f}}_{mn} + \varepsilon^2(\alpha k_1 + \beta k_2 - k_4)/k_3$.

After eliminating the secular terms, the solution of (55)–(57) for this problem can be obtained by letting

$$\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix} = \begin{Bmatrix} U_1 \cos\alpha x \sin\beta y \\ V_1 \sin\alpha x \cos\beta y \\ W_1 \sin\alpha x \sin\beta y \end{Bmatrix} W_0 \cos\{\omega^{\mathrm{f}}_{mn}\tau_0 + [(\alpha k_1 + \beta k_2 - k_4)/k_3]\tau_1 + \tilde{\delta}_{mn}\}, \tag{82}$$

where $\tilde{\delta}_{mn}$ are functions of $\tau_2, \tau_3, \ldots$.

Substituting (82) into (55)–(57), we obtain

$$[A_{11}\alpha^2 + A_{66}\beta^2 - I_{10}(\omega^{\mathrm{f}}_{mn})^2]U_1 + (A_{12} + A_{66})\alpha\beta V_1 = k_1, \tag{83}$$

$$(A_{12} + A_{66})\alpha\beta U_1 + [A_{66}\alpha^2 + A_{22}\beta^2 - I_{10}(\omega^{\mathrm{f}}_{mn})^2]V_1 = k_2, \tag{84}$$

$$\{D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4 - [I_{20} + I_{12}(\alpha^2 + \beta^2)](\omega^{\mathrm{f}}_{mn})^2\}W_1 = 0. \tag{85}$$

Obviously, with (67), eqn (85) is identically satisfied and (83) and (84) permit a unique solution for U_1 and V_1. Upon using (82) in (51)–(54) and (30), the ε^2 corrections to the displacements and stresses can be determined in a way similar to the first-order solution. The results for displacements are

$$u_{(1)} = [U_1 - \alpha z W_1 + \tilde{\Phi}_{11}(z)] W_0 \cos \alpha x \sin \beta y \cos \{\omega^{\rm f}_{mn}\tau_0 + [(\alpha k_1 + \beta k_2 - k_4)/k_3]\tau_1 + \tilde{\delta}_{mn}\}, \tag{86}$$

$$v_{(1)} = [V_1 - \beta z W_1 + \tilde{\Phi}_{21}(z)] W_0 \sin \alpha x \cos \beta y \cos \{\omega^{\rm f}_{mn}\tau_0 + [(\alpha k_1 + \beta k_2 - k_4)/k_3]\tau_1 + \tilde{\delta}_{mn}\}, \tag{87}$$

$$w_{(1)} = [W_1 + \tilde{\varphi}_1(z)] W_0 \sin \alpha x \sin \beta y \cos \{\omega^{\rm f}_{mn}\tau_0 + [(\alpha k_1 + \beta k_2 - k_4)/k_3]\tau_1 + \tilde{\delta}_{mn}\}, \tag{88}$$

in which $\tilde{\Phi}_{11}(z)$ *and* $\tilde{\Phi}_{21}$ are given in the Appendix.

The second-order solution is obtained by combining (68)–(76) with (86)–(88) using (20). Thus, we have the displacement field given by

$$u = [\varepsilon^2 U_1 - \alpha(1 + \varepsilon^2 W_1)z + \varepsilon^2 \tilde{\Phi}_{11}(z)] W_0 \cos \alpha x \sin \beta y \times \cos \{\omega^{\rm f}_{mn}\tau_0 + [(\alpha k_1 + \beta k_2 - k_4)/k_3]\tau_1 + \tilde{\delta}_{mn}\}, + O(\varepsilon^4), \tag{89}$$

$$v = [\varepsilon^2 V_1 - \beta(1 + \varepsilon^2 W_1)z + \varepsilon^2 \tilde{\Phi}_{21}(z)] W_0 \sin \alpha x \cos \beta y \times \cos \{\omega^{\rm f}_{mn}\tau_0 + [(\alpha k_1 + \beta k_2 - k_4)/k_3]\tau_1 + \tilde{\delta}_{mn}\}, + O(\varepsilon^4), \tag{90}$$

$$w = [1 + \varepsilon^2 W_1 + \varepsilon^2 \tilde{\varphi}_1(z)] W_0 \, sin \, \alpha x \sin \beta y \times \cos \{\omega^{\rm f}_{mn}\tau_0 + [(\alpha k_1 + \beta k_2 - k_4)/k_3]\tau_1 + \tilde{\delta}_{mn}\}, + O(\varepsilon^4). \tag{91}$$

We observe from these expressions that at least for this problem the first-order displacement field for the flexural motion is linear in z as assumed by the classical plate theories. However, the displacements of the ε^2 order are related to $\tilde{\varphi}_1(z)$, $\tilde{\Phi}_{11}(z)$ and $\tilde{\Phi}_{21}(z)$ which depend on the transverse shears as well as the lamination scheme. Evidently the variations of the displacements through the plate thickness are not only loading dependent but also very much material dependent. When the material is homogeneous, the relevant expressions can be integrated to give the z-dependence for the ε^2 order as $\tilde{\varphi}_1(z) \sim z^2$, $\tilde{\Phi}_{11}(z) \sim z; z^2; z^3$ and $\tilde{\Phi}_{21}(z) \sim z; z^2; z^3$. Then the variations of the displacements through the thickness can be represented by simple polynomials in z. The results put the higher-order plate theories in perspective.

In Table 1 we present the numerical results for the fundamental frequency of a square [0/90/90/0] laminated plate in comparison with the results obtained according to CLT, the first-order shear deformation theory (FSDT) and a higher-order shear deformation theory (HSDT) (Reddy and Phan, 1985). We made the computation using the layer material properties given by $E_{\rm L}/E_{\rm T} = 40$, $G_{\rm LT}/E_{\rm T} = 0.6$, $G_{\rm TT}/E_{\rm T} = 0.5$, $\nu_{\rm LT} = \nu_{\rm TT} = 0.25$, where the subscripts L and T refer to the longitudinal direction and transverse to fiber direction. The elastic constants can be deduced from these data. The asymptotic solution was carried out to the second order. In cases where the span-to-thickness ratio is large the second-order modifications are minor. The modifications become significant as the thickness increases

Table 1. Comparisons of the normalized fundamental frequency, $\tilde{\omega} = \omega^{\rm f}_{11} a^2 (\rho_0/E_{\rm T})^{1/2}/2h$, a is the plate dimension

$a/2h$	10	12.5	20	25	50	100
CLT	18.891	18.891	18.891	18.891	18.891	18.891
FSDT	15.083	16.120	17.583	17.991	18.590	18.750
HSDT	15.270	16.276	17.668	18.050	18.606	18.755
Present (ε^0)	18.738	18.793	18.853	18.866	18.885	18.890
Present (ε^2)	13.417	15.349	17.491	17.993	18.665	18.835

due to the pronounced thickness effect. When the span-to-thickness ratio is less than 10 the results of the second-order approximations are not as conformable with the FSDT and HSDT results as we would expect. The asymptotic solution must be continued to the higher order for thick plates. In view of the recurrence of the solution forms [compare (60)–(62) with (82)], the asymptotic solution for this problem in principle can be determined to any order and provides results with any degree of accuracy. However, the expressions of the analytical solution become too involved to be meaningful. We shall not pursue the asymptotic analysis further at this time.

6. CONCLUDING REMARKS

The equations for the asymptotic solution based on the present theory are essentially the same as the CLT equations. However, the CLT equations in the general forms are three simultaneous partial differential equations which are very difficult to solve analytically, even in the case of elastostatics. The solution is less formidable when the plate is symmetric about the midplane as the equations are uncoupled. Analytical solutions for a few problems in which the equations are coupled may be obtained. For example, the problems of antisymmetric cross-ply and angle-ply laminated plates with appropriate edge conditions can be solved analytically with relative ease using the present formulation. The CLT solutions for these problems can be found in Ashton and Whitney (1970). In fact, as long as the CLT solution for a problem is available, we can determine immediately the first-order solution for the displacements and all the stress components, including the transverse shear and normal stresses. In the case of laminated plates the interfacial continuity conditions are inherently satisfied. The advantage is more appreciable when the laminated plates are composed of many layers in that there is no need to treat the layers individually.

When the flexural motion and the in-plane motion are coupled together, it is essential to distinguish from the simultaneous equations at the higher-order level which terms cause the secular behavior in the particular solution and eliminate them accordingly. They can be easily identified provided that the equations are decoupled. The decoupling can be accomplished by expressing the displacements in terms of the normal modes of vibration, using the standard procedure [see, e.g. Meirovitch (1967)] for vibration analysis of multi-degrees-of-freedom systems.

In the illustrative example we considered only the free vibration characteristics of the laminated plate. The transient response due to forced motion in itself is complicated. The eigenfunction expansions of the field variables in terms of the normal modes of the plates may be employed. Alternatively, the Newmark's direct integration method may be used as exemplified in Reddy (1982). Although the first-order solution is relatively easy to obtain, the expressions for the higher-order solution are more involved. Therefore, it is essential to have a numerical scheme which enables one to calculate the higher-order corrections without determining the lengthy expressions for the field quantities explicitly when necessary data of a problem are provided. Development of a numerical scheme in conjunction with the asymptotic theory is of practical importance and is the subject of continuing study.

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APPENDIX

The relevant functions for the ε^2 solution of the example problem are

$$\varphi_1 = \tilde{\varphi}_1(z) W_0 \cos(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn}) \sin\alpha x \sin\beta y, \tag{A1}$$

$$\phi_{11} = \tilde{\phi}_{11}(z) W_0 \cos(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn}) \cos\alpha x \sin\beta y, \tag{A2}$$

$$\phi_{21} = \tilde{\phi}_{21}(z) W_0 \cos(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn}) \sin\alpha x \cos\beta y, \tag{A3}$$

$$f_{11} = \left[\tilde{k}_1(z) + 2\alpha\omega^{\mathrm{f}}_{mn}\left(\int_{-1}^{z} \rho_1 z \,\mathrm{d}z\right)\frac{\partial \delta^{\mathrm{f}}_{mn}}{\partial \tau_1}\right] W_0 \cos(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn}) \cos\alpha x \sin\beta y, \tag{A4}$$

$$f_{21} = \left[\tilde{k}_2(z) + 2\beta\omega^{\mathrm{f}}_{mn}\left(\int_{-1}^{z} \rho_1 z \,\mathrm{d}z\right)\frac{\partial \delta^{\mathrm{f}}_{mn}}{\partial \tau_1}\right] W_0 \cos(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn}) \sin\alpha x \cos\beta y, \tag{A5}$$

$$q_1 = \left[\tilde{k}_3(z)\frac{\partial \delta^{\mathrm{f}}_{mn}}{\partial \tau_1} + \tilde{k}_4(z)\right] W_0 \cos(\omega^{\mathrm{f}}_{mn}\tau_0 + \delta^{\mathrm{f}}_{mn}) \sin\alpha x \sin\beta y, \tag{A6}$$

in which

$$\tilde{\varphi}_1(z) = -\int_0^z (c_{13}\alpha^2 + c_{23}\beta^2) c_{33}^{-1} z \,\mathrm{d}z,$$

$$\tilde{\phi}_{11}(z) = \int_0^z [\tilde{s}_{55}\sigma_{xz0} + (z-\eta)(c_{13}\alpha^2 + c_{23}\beta^2) c_{33}^{-1}\alpha\eta] \,\mathrm{d}\eta,$$

$$\tilde{\phi}_{21}(z) = \int_0^z [\tilde{s}_{44}\sigma_{yz0} + (z-\eta)(c_{13}\alpha^2 + c_{23}\beta^2) c_{33}^{-1}\beta\eta] \,\mathrm{d}\eta,$$

$$\tilde{f}_{11}(z) = [\tilde{Q}_{11}\alpha^2 + \tilde{Q}_{66}\beta^2 - \rho_1(\omega^{\mathrm{f}}_{mn})^2]\tilde{\phi}_{11} + (\tilde{Q}_{12} + \tilde{Q}_{66})\alpha\beta\tilde{\phi}_{21} - \alpha\sigma_{z0}c_{13}c_{33}^{-1},$$

$$\tilde{f}_{21}(z) = (\tilde{Q}_{12} + \tilde{Q}_{66})\alpha\beta\tilde{\phi}_{11} + [\tilde{Q}_{66}\alpha^2 + \tilde{Q}_{22}\beta^2 - \rho_1(\omega^{\mathrm{f}}_{mn})^2]\tilde{\phi}_{21} - \beta\sigma_{z0}c_{23}c_{33}^{-1},$$

$$\tilde{k}_1(z) = \int_{-1}^{z} \tilde{f}_{11}(z)\,\mathrm{d}z, \quad \tilde{k}_2(z) = \int_{-1}^{z} \tilde{f}_{21}(z)\,\mathrm{d}z,$$

$$\tilde{k}_3(z) = 2\omega^{\mathrm{f}}_{mn}\left[(\alpha^2+\beta^2)\int_{-1}^{z} \rho_1\eta(z-\eta)\,\mathrm{d}\eta - \int_{-1}^{z} \rho_2\,\mathrm{d}\eta\right],$$

$$\tilde{k}_4(z) = \int_{-1}^{z} (z-\eta)(\alpha\tilde{f}_{11} + \beta\tilde{f}_{21})\,\mathrm{d}\eta - (\omega^{\mathrm{f}}_{mn})^2 \int_{-1}^{z} \rho_2\tilde{\varphi}_1(\eta)\,\mathrm{d}\eta.$$

At $z = 1$, $I_{11} = \int_{-1}^{1} \rho_1 z\,\mathrm{d}z = 0$ because of symmetry, then we have

$$k_1 = \int_{-1}^{1} \tilde{f}_{11}(z)\,\mathrm{d}z, \quad k_2 = \int_{-1}^{1} \tilde{f}_{21}(z)\,\mathrm{d}z, \tag{A7,8}$$

$$k_3 = -2\omega^{\mathrm{f}}_{mn}[(\alpha^2+\beta^2)I_{12} + I_{20}], \tag{A9}$$

$$k_4 = \int_{-1}^{1} (1-z)(\alpha\tilde{f}_{11} + \beta\tilde{f}_{21})\,\mathrm{d}z - (\omega^{\mathrm{f}}_{mn})^2 \int_{-1}^{1} \rho_2\tilde{\varphi}_1(z)\,\mathrm{d}z. \tag{A10}$$